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ORIGINAL ARTICLE

Similarity solutions of quasi three dimensional power law fluids using the method of satisfaction of asymptotic boundary conditions

Nita Jain [a,*], **M.G. Timol** [b]

[a] *Department of Humanities & Sciences, Thakur College of Engineering and Technology, Mumbai 101, Maharashtra, India*
[b] *Department of Mathematics, Veer Narmad South Gujarat University, Surat 7, Gujarat, India*

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Abstract The work presented in this paper is focused on deductive group-theoretic transformations to develop the similarity solution of steady, laminar, incompressible quasi three dimensional boundary layer flow governing power law fluid. The application of one-parameter group reduces the number of independent variables to one and consequently the system of governing, highly non-linear partial differential equations reduces to a self similar, non-linear ordinary differential equation with appropriate auxiliary conditions. The numerical solution for a power law fluid considered for small cross flow is obtained systematically using MSABC in dimensionless form.

1. Introduction

The boundary layer flow occurs over various aerodynamic configurations such as wings, missiles, fuselage forms or channel flows. For the flow over wings and channels, the boundary layer generally develops along a line on the surface (e.g. leading edge). The development of boundary layer is useful for efficient design procedures of both internal turbo machine components and external surface components. The gross allowances for the boundary layer were possible to make, using empirical relations along with results predicted from two dimensional boundary layer theory. But this method does not seem much promising for three dimensional boundary layer characteristics. As a result, a great deal of experimental and theoretic research on three dimensional flows has been carried on in recent years. But still a need exists for theoretic analyses which will lead to the predictions of boundary layer behavior.

Theoretical research has been restricted because of the complex nature of the equations describing the flow, while in a practical problems, the boundary layer is turbulent. Some success, however, has been achieved in the analysis of the three dimensional laminar equations. There is always a need to search for exact solutions of the laminar, incompressible, boundary layer equations for special types of main stream flows. From a physical point of view, these boundary layer equations have the capacity to admit a large number of invariant solutions which are known as similarity solutions. These invariant solutions are meant to reduce nonlinear partial differential equations of the boundary layer to a system of ordinary differential equations.

* Corresponding author.
E-mail address: nita.jain109@gmail.com (N. Jain).

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Nomenclature

u, v, w	velocity component in the boundary layer along x, y, z-axis respectively
τ_{yx}, τ_{yz}	the two non-vanishing components of the shear tensor
x, y	Cartesian coordinates
m, n	parameters in the mathematical model of a power-law fluid
U, W	velocity component in the main flow along x, z-axis respectively
ρ	density of the fluid
L, U_0	characteristic length and velocity respectively
Re	Reynolds number
ψ	arbitrary mathematical function
G, $\overline{G}$, g	arbitrary constants
Π, €, £	arbitrary constants
$N_1 \ldots N_4$	arbitrary constants
a, $\propto_{11} \ldots \propto_{81}$	arbitrary constants
β, $\beta_{11} \ldots \beta_{81}$	arbitrary constants
η	independent variable in transformed ordinary differential equation
$F_1 \ldots F_4$	dependent variables in transformed ordinary differential equation

For any non-Newtonian fluid, two entities are important viz. the mathematical structure of the shearing stress and the rate of shear. Such a mathematical formulation is indeed a difficult task. Since great diversity is found in the physical structure of non-Newtonian fluids, it is difficult to recommend a single constitutive equation to describe them. When shear stress is an arbitrary function of the velocity gradient, the non-Newtonian fluid model represents Visco-Inelastic behavior which is observed in several fluids including Newtonian fluids, Prandtl fluids, Prandtl–Eyring fluids, Power-law fluids, Eyring fluids, Sisko fluids, Sutterby fluids, Ellis fluids, Williamson fluids, Reiner–Philippoff fluids, Powell Eyring fluids. To investigate the non-Newtonian effects, similarity solutions play an important role because being exact solutions, they serve as a reference to check approximate solutions.

Hansen and Herzig [1–3] has developed three dimensional boundary layer equations for the flow past flat surface for Cartesian, Curvilinear and Polar coordinate system and derived similarity solution for all three cases. But all these past cases were limited to Newtonian fluids only. Schowalter [4] was probably the first to introduced similarity solution for three dimensional non-Newtonian Power law fluids using separation of variable method. Na and Hansen [5] have obtained similarity solution for three dimensional boundary layer equations for non-Newtonian Power law fluids by linear and spiral group of transformations. They have also obtained similarity solution for small cross flow geometry. Timol and Kalthia [6] have carried out similarity analysis of three dimensional boundary layer equations of non-Newtonian fluids and integrated similarity equations for Reiner Philippoff fluids. Pakdemirli [7,8] and his co-workers have worked out similarity solutions for three dimensional boundary layer flow of non-Newtonian Power law fluids using scaling and spiral group of transformations. However, in all these cases, similarity equations were highly nonlinear coupled differential equations and remain numerically unsolved.

Nowadays, many techniques are available for similarity analysis. Among them, the similarity methods which invoke the invariance under the group of transformations are known as group theoretic methods. These methods are more recent and are mathematically elegant; hence they are widely used in different fields. The group theoretic methods involve mainly two different types of groups of transformations, namely, assumed group of transformations and deductive group of transformations. The linear group transformations, scaling group transformations, spiral group transformations are the assumed group of transformations and are mainly due to Birkhoff [9] and Morgan [10] where as the deductive group of transformations can be further classified into two groups: finite group of transformations Moran and Gaggioli [11] and infinitesimal group of transformation Bluman and Cole [12], Bluman and Kumai [13].

The main drawback of similarity methods based on the assumed group of transformation at the outset of the analysis is that, the resulting similarity solutions are restrictive and may sometimes lead to wrong conclusion that the similarity transformations does not exist. On the other hand, the similarity methods based on general group of transformation are more systematic and lead to a number of similarity solutions. Out of these, the deductive group theoretic method provides a powerful tool because it is not based on linear operators, superposition, or any other aspect of linear solution techniques. Therefore, this method can successfully be applied to nonlinear differential models.

Recently, deductive group of transformation has been successfully applied to various non-linear two dimensional flow problems by Abd-el-Malek et al. [14], Parmar and Timol [15], Adnan et al. [16] and Darji and Timol [17]. The objective of present investigation was to apply the deductive group method based on general group of transformation to derive similarity solutions for steady, quasi three dimensional incompressible laminar boundary layer flows of non-Newtonian Power law fluid. We treat Ostwald-de model of Power law fluids as it is most widely used model to exhibit non-Newtonian behavior in fluids and to predict shear thinning and shear thickening behavior.

Solution of the final similarity equations in general, requires the application of numerical techniques. Little work has been done on solving these equations. Thus we have made an attempt to find the numerical solution of nonlinear coupled ordinary differential equations.

2. Governing equations

The governing differential equations for the boundary layer flow of the generalized non-Newtonian fluid are given as [5]

Continuity $\frac{\partial u'}{\partial x'}+\frac{\partial v'}{\partial y'}=0$ (1)

Momentum $u'\frac{\partial u'}{\partial x'}+v'\frac{\partial u'}{\partial y'}=\frac{1}{\rho}\frac{\partial \tau'_{y'x'}}{\partial y'}+U'\frac{\partial U'}{\partial x'}$ (2)

$$u'\frac{\partial w'}{\partial x'}+v'\frac{\partial w'}{\partial y'}=\frac{1}{\rho}\frac{\partial \tau'_{y'z'}}{\partial y'}+U'\frac{\partial w'}{\partial x'} \tag{3}$$

Shear stress $\tau'_{y'x'}=-m\left[\left(\frac{\partial u'}{\partial y'}\right)^2+\left(\frac{\partial w'}{\partial y'}\right)^2\right]^{\frac{n-1}{2}}\frac{\partial u'}{\partial y'}$ (4)

$$\tau'_{y'z'}=-m\left[\left(\frac{\partial u'}{\partial y'}\right)^2+\left(\frac{\partial w'}{\partial y'}\right)^2\right]^{\frac{n-1}{2}}\frac{\partial w'}{\partial y'} \tag{5}$$

With the boundary conditions:

$u'=v'=w'=0,\ \text{at}\ y'=0$ (6.a)

$u'=U'(x'), w'=W'(x')\ \text{at}\ y'\to\infty$ (6.b)

where $\tau'_{y'x'}$ and $\tau'_{y'z'}$ are shearing stresses to Y-direction and acting along X and Z direction respectively.

According to Timol and Timol [18], we have also considered the flow past a semi infinite flat plate placed in the direction of flow. The plate is in X–Z direction and is placed between $0\leqslant x<\infty$ and $-\infty<z<\infty$ and free stream is in the X-direction. The considered flow problem is quasi-three dimensional in nature since the velocity components are independent of the z-coordinates and the stream lines for such flows form a system of 'translates'. It is hoped that by assuming independence of flow quantities in one direction, more quantitative information may be obtained on the characteristics of three dimensional boundary layer flows.

3. Formulation of the problem

Now considering the dimensionless quantities in Eqs. (1)–(6):

$$x=\frac{x'}{L}\quad u=\frac{u'}{U_0}\quad w=\frac{w'}{U_0}\quad U=\frac{U'}{U_0}\quad W=\frac{W'}{U_0}$$

$$y=R_e^{\frac{1}{n+1}}\frac{y'}{L}\quad v=R_e^{\frac{1}{n+1}}\frac{v'}{U_0}\quad R_e=\frac{\rho L^n U_0^{2-n}}{\mu}$$

$$\tau_{yx}=\frac{\tau'_{y'x'}R_e^{\frac{1}{n+1}}}{-m\rho U_0^2}\quad \tau_{yz}=\frac{\tau'_{y'z'}R_e^{\frac{1}{n+1}}}{-m\rho U_0^2}$$

and introducing the stream function ψ as $u=\frac{\partial\psi}{\partial y}$ and $v=-\frac{\partial\psi}{\partial x}$, continuity equation gets satisfied automatically.

$$\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y}-\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2}-\frac{\partial\tau_{yx}}{\partial y}-U\frac{dU}{dx}=0 \tag{7}$$

$$\frac{\partial\psi}{\partial y}\frac{\partial w}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial w}{\partial y}-\frac{\partial\tau_{yz}}{\partial y}-U\frac{dW}{dx}=0 \tag{8}$$

subject to the boundary conditions:

$\frac{\partial\psi}{\partial y}=\frac{\partial\psi}{\partial x}=w=0\ \text{at}\ y=0$ (9.a)

$\frac{\partial\psi}{\partial y}=U(x), w=W(x)\ \text{at}\ y\to\infty$ (9.b)

$$\tau_{yx}=\left[\left(\frac{\partial^2\psi}{\partial y^2}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2\right]^{\frac{n-1}{2}}\frac{\partial^2\psi}{\partial y^2} \tag{10}$$

$$\tau_{yz}=\left[\left(\frac{\partial^2\psi}{\partial y^2}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2\right]^{\frac{n-1}{2}}\frac{\partial w}{\partial y} \tag{11}$$

Eqs. (7)–(11) represent a system of non-linear partial differential equations, the solution of which is quite difficult to obtain. One major simplification can be achieved using similarity transformations where the system of non-linear partial differential equations is reduced to a system of ordinary differential equations.

4. Methodology and solution of the problem

We now seek some sort of transformation namely, similarity transformation which transforms the partial differential Eqs. (7) and (8) into the ordinary differential equations along with appropriate auxiliary conditions. To search this transformation we applied one parameter general deductive group of transformation.

4.1. The group systematic formulation

Introducing the group theoretic method

$G:\Pi_\varepsilon(@)=€^{@}(\varepsilon)@+£^{@}(\varepsilon)$ (12)

where @ stands for x, y, ψ, w, τ_{yx}, τ_{yz}, U, W.

Here $€'s$ and $£'s$ are real-valued and are at least differentiable in the real argument ε.

4.2. The invariance analysis

Eqs. (7)–(11) are invariantly transformed for some functions $N_i(\varepsilon)$ whenever

$$\frac{\partial\overline{\psi}}{\partial\overline{y}}\frac{\partial^2\overline{\psi}}{\partial\overline{x}\partial\overline{y}}-\frac{\partial\overline{\psi}}{\partial\overline{x}}\frac{\partial^2\overline{\psi}}{\partial\overline{y}^2}-\frac{\partial\overline{\tau_{yx}}}{\partial\overline{y}}-\overline{U}\frac{d\overline{U}}{d\overline{x}} =N_1(\varepsilon)\left(\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y}-\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2}-\frac{\partial\tau_{yx}}{\partial y}-U\frac{\mathrm{dU}}{\mathrm{dx}}\right) \tag{13}$$

$$\frac{\partial\overline{\psi}}{\partial\overline{y}}\frac{\partial\overline{w}}{\partial\overline{x}}-\frac{\partial\overline{\psi}}{\partial\overline{x}}\frac{\partial\overline{w}}{\partial\overline{y}}-\frac{\partial\overline{\tau_{yz}}}{\partial\overline{y}}-\overline{U}\frac{d\overline{W}}{d\overline{x}} =N_2(\varepsilon)\left(\frac{\partial\psi}{\partial y}\frac{\partial w}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial w}{\partial y}-\frac{\partial\tau_{yz}}{\partial y}-U\frac{dW}{dx}\right) \tag{14}$$

$\overline{\tau_{yx}}=N_3(\varepsilon)\tau_{yx}$ and $\overline{\tau_{yz}}=N_4(\varepsilon)\tau_{yz}$ (15)

Under the one parameter group transformation (12) and applying chain rule for transforming the derivatives, the above Eqs. (13)–(15) become

$$\left(\frac{€^{\psi}}{€^{y}}\right)\frac{\partial\psi}{\partial y}\left(\frac{€^{\psi}}{€^{x}€^{y}}\right)\frac{\partial^2\psi}{\partial x\partial y}-\left(\frac{€^{\psi}}{€^{x}}\right)\frac{\partial\psi}{\partial x}\left(\frac{€^{\psi}}{€^{y^2}}\right)\frac{\partial^2\psi}{\partial y^2} -\left(\frac{€^{\tau_{yx}}}{€^{y}}\right)\frac{\partial\tau_{yx}}{\partial y}-(€^{U}U+£^{U})\left(\frac{€^{U}}{€^{x}}\right)\frac{dU}{dx} =N_1(\varepsilon)\left(\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y}-\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2}-\frac{\partial\tau_{yx}}{\partial y}-U\frac{dU}{dx}\right) \tag{16}$$

$$\left(\frac{€^{\psi}}{€^{y}}\right)\frac{\partial\psi}{\partial y}\left(\frac{€^{w}}{€^{x}}\right)\frac{\partial w}{\partial x}-\left(\frac{€^{\psi}}{€^{x}}\right)\frac{\partial\psi}{\partial x}\left(\frac{€^{w}}{€^{y}}\right)\frac{\partial w}{\partial y}-\left(\frac{€^{\tau_{yz}}}{€^{y}}\right)\frac{\partial\tau_{yz}}{\partial y}$$
$$-\left(€^{U}U+£^{U}\right)\left(\frac{€^{W}}{€^{x}}\right)\frac{dW}{dx}$$
$$=N_2(\varepsilon)\left(\frac{\partial\psi}{\partial y}\frac{\partial w}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial w}{\partial y}-\frac{\partial\tau_{yz}}{\partial y}-U\frac{dW}{dx}\right) \tag{17}$$

$$\left[\left(\frac{€^{\psi}}{€^{y^2}}\frac{\partial^2\psi}{\partial y^2}\right)^2+\left(\frac{€^{w}}{€^{y}}\frac{\partial w}{\partial y}\right)^2\right]^{\frac{n-1}{2}}\left(\frac{€^{\psi}}{€^{y^2}}\right)\frac{\partial^2\psi}{\partial y^2}$$
$$=N_3(\varepsilon)\left[\left(\frac{\partial^2\psi}{\partial y^2}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2\right]^{\frac{n-1}{2}}\frac{\partial^2\psi}{\partial y^2} \tag{18}$$

$$\left[\left(\frac{€^{\psi}}{€^{y^2}}\frac{\partial^2\psi}{\partial y^2}\right)^2+\left(\frac{€^{w}}{€^{y}}\frac{\partial w}{\partial y}\right)^2\right]^{\frac{n-1}{2}}\left(\frac{€^{w}}{€^{y}}\right)\frac{\partial w}{\partial y}$$
$$=N_4(\varepsilon)\left[\left(\frac{\partial^2\psi}{\partial y^2}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2\right]^{\frac{n-1}{2}}\frac{\partial w}{\partial y} \tag{19}$$

Eqs. (16)–(19) are transformed invariantly if

$$\frac{€^{\psi^2}}{€^{x}€^{y^2}}=\frac{€^{\tau_{yx}}}{€^{y}}=\frac{€^{U^2}}{€^{y}}=N_1(\varepsilon)\ \text{ and }\ R_1=0\ \text{ where }\ R_1=\frac{£^{U}€^{U}}{€^{x}}\frac{dU}{dx} \tag{20}$$

$$\frac{€^{\psi}€^{w}}{€^{x}€^{y}}=\frac{€^{\tau_{yz}}}{€^{y}}=\frac{€^{U}€^{W}}{€^{x}}=N_2(\varepsilon)\ \text{ and }\ R_2=0\ \text{ where }$$
$$R_2=\frac{£^{U}€^{W}}{€^{x}}\frac{\mathrm{dW}}{\mathrm{dx}} \tag{21}$$

Again the invariance of boundary conditions gives:

$$\text{At}\quad y=0,\ £^{y}=£^{w}=0 \tag{22.a}$$

$$\text{At}\quad y=\infty,\ £^{U}=£^{w}=£^{W}=0\quad\text{and}\quad\frac{€^{\psi}}{€^{y}}=€^{U},€^{w}=€^{W} \tag{22.b}$$

On solving these (20)–(22), we obtained:

$$€^{x}=€^{y^3},\quad €^{\psi}=€^{y^2},\quad €^{w}=€^{U}=€^{W}=€^{y},\quad €^{\tau_{yx}}=€^{\tau_{yz}}=1 \tag{23}$$

$$£^{y}=£^{w}=£^{U}=£^{W}=£^{\tau_{yx}}=£^{\tau_{yz}}=0 \tag{24}$$

Finally, we get the one-parameter group $\overline{G}$, which invariantly transforms the differential equations and the auxiliary conditions (7)–(11) as

$$\overline{G}:\quad\begin{aligned}
&\overline{x}=€^{y^3}(\varepsilon)x+£^{x}(\varepsilon)\\
&\overline{y}=€^{y}(\varepsilon)y\ \ \overline{\psi}=€^{y^2}(\varepsilon)\psi+£^{\psi}(\varepsilon)\\
&\overline{w}=€^{y}(\varepsilon)w\\
&\overline{\tau}_{yx}=\tau_{yx}\\
&\overline{\tau}_{yz}=\tau_{yz}\\
&\overline{U}=€^{y}(\varepsilon)U\\
&\overline{W}=€^{y}(\varepsilon)W
\end{aligned} \tag{25}$$

4.3. The complete set of absolute invariants

Now we want to develop a complete set of absolute invariants so that the original problem (7)–(11) will transformed into similarity equations via group theoretic method. We have applied HAMAD [19] formulations for PDEs of 2-independent variables.

Considering $x_1=x,\ x_2=y,\ y_1=\psi$, $y_2=w,\ y_3=\tau_{yx}$, $y_4=\tau_{yz}$, $y_5=U,$ $y_6=W$ and also $\propto_{i1}=\frac{\partial€^{i}}{\partial\varepsilon}\Big|_{\varepsilon=\varepsilon_0}$ and $\beta_{i1}=\frac{\partial£^{i}}{\partial\varepsilon}\Big|_{\varepsilon=\varepsilon_0}$, $i=1$ to 8,where ε_0 denotes the value of ε which yield the identity element of the group. The generator for eliminating the independent variable is given by

$$\begin{aligned}X=&(\propto_{11}x_1+\beta_{11})\frac{\partial g}{\partial x_1}+(\propto_{21}x_2+\beta_{21})\frac{\partial g}{\partial x_2}\\
&+(\propto_{31}y_1+\beta_{31})\frac{\partial g}{\partial y_1}+(\propto_{41}y_2+\beta_{41})\frac{\partial g}{\partial y_{21}}\\
&+(\propto_{51}y_3+\beta_{51})\frac{\partial g}{\partial y_3}+(\propto_{61}y_4+\beta_{61})\frac{\partial g}{\partial y_4}\\
&+(\propto_{71}y_5+\beta_{71})\frac{\partial g}{\partial y_5}+(\propto_{81}y_6+\beta_{81})\frac{\partial g}{\partial y_6}\end{aligned} \tag{26}$$

Hence the characteristic equation becomes

$$\frac{dx}{\propto_{11}x+\beta_{11}}=\frac{dy}{\propto_{21}y}=\frac{d\psi}{\propto_{31}\psi+\beta_{31}}=\frac{dw}{\propto_{41}w}=\frac{d\tau_{yx}}{0}=\frac{d\tau_{yz}}{0}=\frac{dU}{\propto_{71}U}=\frac{dW}{\propto_{81}W} \tag{27}$$

On integrating the characteristic Eq. (27) using variable separable method, we obtain absolute invariants for independent and dependent variables as follows:

$$\eta=y\pi^{-\frac{\propto_{21}}{\propto_{11}}}\ \text{where}\ \pi=x+\beta\ \text{and}\ \beta=\frac{\beta_{11}}{\propto_{11}}$$

$$\psi=\pi^{\frac{\propto_{31}}{\propto_{11}}}\quad F_1(\eta)-\frac{\beta_{31}}{\propto_{31}}\qquad w=\pi^{\frac{\propto_{41}}{\propto_{11}}}F_2(\eta) \tag{28}$$

$$\tau_{yx}=\tau_{yx}\quad\tau_{yz}=\tau_{yz}\quad U=\pi^{\frac{\propto_{71}}{\propto_{11}}}F_3(\eta)\quad W=\pi^{\frac{\propto_{81}}{\propto_{11}}}F_4(\eta)$$

Substituting (28) in Eqs. (7), (8) and (10)

$$\begin{aligned}&\left(\pi^{\frac{\propto_{31}-\propto_{21}}{\propto_{11}}}F_1'\right)\pi^{\frac{\propto_{31}-\propto_{21}}{\propto_{11}}-1}\left(\frac{\propto_{31}-\propto_{21}}{\propto_{11}}F_1'-\frac{\propto_{21}}{\propto_{11}}\eta F_1''\right)\\
&-\pi^{\frac{\propto_{31}}{\propto_{11}}-1}\left(\frac{\propto_{31}}{\propto_{11}}F_1-\frac{\propto_{21}}{\propto_{11}}\eta F_1'\right)\left(\pi^{\frac{\propto_{31}-2\propto_{21}}{\propto_{11}}}F_1''\right)-\pi^{-\frac{\propto_{21}}{\propto_{11}}}\tau_{yx}'\\
&-\pi^{\frac{2\propto_{71}}{\propto_{11}}-1}\frac{\propto_{71}}{\propto_{11}}F_3F_3'=0\end{aligned} \tag{29}$$

$$\begin{aligned}&\left(\pi^{\frac{\propto_{31}-\propto_{21}}{\propto_{11}}}F_1'\right)\pi^{\frac{\propto_{41}}{\propto_{11}}-1}\left(\frac{\propto_{41}}{\propto_{11}}F_2-\frac{\propto_{21}}{\propto_{11}}\eta F_2'\right)\\
&-\pi^{\frac{\propto_{31}}{\propto_{11}}-1}\left(\frac{\propto_{31}}{\propto_{11}}F_1-\frac{\propto_{21}}{\propto_{11}}\eta F_1'\right)\left(\pi^{\frac{\propto_{41}-\propto_{21}}{\propto_{11}}}F_2'\right)-\pi^{-\frac{\propto_{21}}{\propto_{11}}}\tau_{yz}'\\
&-\pi^{\frac{\propto_{71}+\propto_{81}}{\propto_{11}}-1}\frac{\propto_{81}}{\propto_{11}}F_3F_4'=0\end{aligned} \tag{30}$$

$$\tau_{yx}=\left[\left(\pi^{\frac{\propto_{31}-2\propto_{21}}{\propto_{11}}}F_1''\right)^2+\left(\pi^{\frac{\propto_{41}-\propto_{21}}{\propto_{11}}}F_2'\right)^2\right]^{\frac{n-1}{2}}\pi^{\frac{\propto_{31}-2\propto_{21}}{\propto_{11}}}F_1'' \tag{31}$$

$$\tau_{yz} = \left[\left(\pi^{\frac{\propto_{31}-2\propto_{21}}{\propto_{11}}} F_1''\right)^2 + \left(\pi^{\frac{\propto_{41}-\propto_{21}}{\propto_{11}}} F_2'\right)^2\right]^{\frac{n-1}{2}} \pi^{\frac{\propto_{41}-\propto_{21}}{\propto_{11}}} F_2' \tag{32}$$

For Eqs. (29)–(32) are reduced to a system of ordinary differential equations, it is necessary that the coefficients should be constants or functions of η only. Thus

$$\frac{2\propto_{31}-2\propto_{21}}{\propto_{11}} - 1 = -\frac{\propto_{21}}{\propto_{11}} = \frac{2\propto_{71}}{\propto_{11}} - 1 \tag{33}$$

$$\frac{\propto_{31}-\propto_{21}+\propto_{41}}{\propto_{11}} - 1 = -\frac{\propto_{21}}{\propto_{11}} = \frac{\propto_{71}+\propto_{81}}{\propto_{11}} - 1 \tag{34}$$

$$\left(\frac{2\propto_{31}-4\propto_{21}}{\propto_{11}}\right)\left(\frac{n-1}{2}\right) + \frac{2\propto_{31}-2\propto_{21}}{\propto_{11}} = 0 = \left(\frac{2\propto_{41}-2\propto_{21}}{\propto_{11}}\right)\left(\frac{n-1}{2}\right)\frac{2\propto_{31}-2\propto_{21}}{\propto_{11}} \tag{35}$$

Solutions of (33)–(35) then gives

$$\frac{\propto_{21}}{\propto_{11}} = \frac{1}{3}\frac{\propto_{31}}{\propto_{11}} = \frac{2}{3}\frac{\propto_{41}}{\propto_{11}} = \frac{\propto_{71}}{\propto_{11}} = \frac{\propto_{81}}{\propto_{11}} = \frac{1}{3} \tag{36}$$

Substituting (36) into (28) we derive similarity variables as follows:

$$\eta = y(x+\beta)^{-\frac{1}{3}} \quad \text{where } \beta = \frac{\beta_{11}}{\propto_{11}}$$

$$\psi = (x+\beta)^{\frac{2}{3}} F_1(\eta) - \frac{\beta_{31}}{\propto_{31}} \quad w = (x+\beta)^{\frac{1}{3}} F_2(\eta) \tag{37}$$

$$\tau_{yx} = \tau_{yx} \quad \tau_{yz} = \tau_{yz} \quad U = (x+\beta)^{\frac{1}{3}} F_3(\eta) \quad W = (x+\beta)^{\frac{1}{3}} F_4(\eta)$$

It should be noted that for the existence of similarity solutions, the considered boundary conditions should be constant. This is possible only when the main flow streamlines are straight lines. Thus, without loss of generality we take $F_3(\eta) = F_4(\eta) = 1$.

4.4. Reduction to an ordinary differential equation

The derived similarity transformations (37) for independent and dependent variables are applied to Eqs. (7)–(11) which results into the following non-linear coupled ordinary differential equations:

$$F_1'^2 - 2F_1F_1'' - 1 = 3\frac{d\tau_{yx}}{d\eta} \tag{38}$$

$$F_1'F_2 - 2F_1F_2' - 1 = 3\frac{d\tau_{yz}}{d\eta} \tag{39}$$

$$\tau_{yx} = \left(F_1''^2 + F_2'^2\right)^{\frac{n-1}{2}} F_1'' \tag{40}$$

$$\tau_{yz} = \left(F_1''^2 + F_2'^2\right)^{\frac{n-1}{2}} F_2' \tag{41}$$

$$F_1(0) = 0, \quad F_1'(0) = 0, \quad F_2(0) = 0 \tag{42.a}$$

$$F_1'(\infty) = 1, \quad F_2(\infty) = 1 \tag{42.b}$$

These equations are exactly similar to those of Patel and Timol [20].

5. Numerical solution for small cross section

Following Hansen and Na [5], here we have considered the small cross flow to obtain numerical solution of (38)–(42)

i.e. $\dfrac{\partial w}{\partial y} \ll \dfrac{\partial u}{\partial y}$

In this case shear stress (40) and (41) becomes

$$\tau_{yx} = F_1'''^{n} \quad \text{and} \quad \tau_{yz} = F_1'''^{n-1} F_2' \tag{43}$$

On substituting (43) in (38) and (39) we derived the system of non-linear ordinary differential equations:

$$F_1''' = \frac{1}{3nF_1'''^{n-1}}\left(F_1'^2 - 2F_1F_1'' - 1\right) \tag{44}$$

$$F_2'' = \frac{1}{3F_1'''^{n-1}}\left(F_1'F_2 - 2F_1F_2' - 1 - 3(n-1)F_1'''^{n-2}F_1'''F_2'\right) \tag{45}$$

The numerical method applied to solve Eqs. (44) and (45) with the boundary condition (42) is the Adams–Moulton procedure along with shooting method due to Nachtsheim and Swigert [21], based on the least square convergence criterion. The asymptotic boundary conditions are satisfied at the edge of the boundary layer by adjusting the initial conditions so that the mean square error between the computed variables and asymptotic values is minimized. The method can be applicable to study the relationship of the properties of solution with the parameter variation.

The method is briefly described below:

Since there are two asymptotic boundary conditions to satisfy, two additional initial conditions at the wall have to be adjusted. So the considered boundary value problem is equivalent to the problem of finding the values of $F_1''(0)$ & $F_2'(0)$ for which the boundary condition at the edge of the boundary layer is satisfied. It means that the solution of the following simultaneous non-linear Eqs. (46) and (47) is to be determined.

$$F'_{1\text{edge}}\left[F_1''(0), F_2'(0)\right] = 1 \tag{46}$$

$$F_{2\text{edge}}\left[F_1''(0), F_2'(0)\right] = 1 \tag{47}$$

where $F'_{1\text{edge}} = F_1'(\eta_{\text{edge}})$ and $F_{2\text{edge}} = F_2(\eta_{\text{edge}})$. Denoting $F_1''(0) = x$ and $F_2'(0) = z$ and following the asymptotic boundary conditions method,

$$F_1''' = \frac{1}{3nF_1'''^{n-1}}\left(F_1'^2 - 2F_1F_1'' - 1\right) \tag{48}$$

$$F_2'' = \frac{1}{3F_1'''^{n-1}}\left(F_1'F_2 - 2F_1F_2' - 1 - 3(n-1)F_1'''^{n-2}F_1'''F_2'\right) \tag{49}$$

$$F_1(0) = 0, F_1'(0) = 0, F_1''(0) = x \tag{50.a}$$

$$F_2(0) = 0, F_2'(0) = z \tag{50.b}$$

Following the principal of least squares, Δx and Δy must be found from the solution of the following matrix equation:

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}\begin{pmatrix} \Delta x \\ \Delta z \end{pmatrix} = -\begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \tag{51}$$

where

$$A_{11} = F_{1x}'^2 + F_{2x}^2 + F_{1x}''^2 + F_{2x}'^2$$
$$A_{12} = A_{21} = F_{1x}'F_{1z}' + F_{2x}F_{2z} + F_{1x}''F_{1z}'' + F_{2x}'F_{2z}'$$
$$A_{22} = F_{1z}'^2 + F_{2z}^2 + F_{1z}''^2 + F_{2z}'^2$$
$$B_1 = F_1'F_{1x}' - F_{1x}' + F_2F_{2x} - F_{2x} + F_1''F_{1x}'' + F_2'F_{2x}'$$
$$B_2 = F_1'F_{1z}' - F_{1z}' + F_2F_{2z} - F_{2z} + F_1''F_{1z}'' + F_2'F_{2z}'$$

The error E between the asymptotic conditions and the computed values at $\eta = \eta_{\text{stop}}$ is given by

$$E = F_1'^2 + F_2^2 + F_1''^2 + F_2'^2 \tag{52}$$

The partial derivatives with respect to x and z that appear in Eq. (51) are obtained by integrating the appropriate perturbation differential equations. The perturbation differential equations for the x derivatives are as follows

$$F_{1x}''' = \frac{1}{3nF_1''^{n-1}}\left(2F_1'F_{1x}' - 2F_{1x}F_1'' - 2F_1F_{1x}'' - 3n(n-1)F_1''^{n-2}F_{1x}''F_1'''\right) \tag{53}$$

$$\begin{aligned} F_{2x}'' = {} & \frac{1}{3F_1''^{n-1}}\left(F_{1x}'F_2 + F_1'F_{2x} - 2F_{1x}F_2' - 2F_1F_{2x}'\right) \\ & - \frac{(n-1)}{F_1''^{n-1}}(F_1''^{n-2}F_1'''F_{2x}' + F_1''^{n-2}F_{1x}'''F_2' \\ & + (n-2)F_1''^{n-3}F_{1x}''F_1'''F_2' + F_1''^{n-2}F_{1x}''F_2'') \end{aligned} \tag{54}$$

along with initial conditions

$$F_{1x}(0) = 0, F_{1x}'(0) = 0, F_{1x}''(0) = 1 \tag{55.a}$$

$$F_{2x}(0) = 0, F_{2x}'(0) = 0 \tag{55.b}$$

The perturbation differential equations for the z derivatives are

$$F_{1z}''' = \frac{1}{3nF_1''^{n-1}}\left(2F_1'F_{1z}' - 2F_{1z}F_1'' - 2F_1F_{1z}'' - 3n(n-1)F_1''^{n-2}F_{1z}''F_1'''\right) \tag{56}$$

$$\begin{aligned} F_{2z}'' = {} & \frac{1}{3F_1''^{n-1}}\left(F_{1z}'F_2 + F_1'F_{2z} - 2F_{1z}F_2' - 2F_1F_{2z}'\right) \\ & - \frac{(n-1)}{F_1''^{n-1}}(F_1''^{n-2}F_1'''F_{2z}' + F_1''^{n-2}F_{1z}'''F_2' \\ & + (n-2)F_1''^{n-3}F_{1z}''F_1'''F_2' + F_1''^{n-2}F_{1z}''F_2'') \end{aligned} \tag{57}$$

along with initial conditions

$$F_{1z}(0) = 0, F_{1z}'(0) = 0, F_{1z}''(0) = 0 \tag{58.a}$$

$$F_{2z}(0) = 0, F_{2z}'(0) = 1 \tag{58.b}$$

Assuming $x = z = 0$, the three system of Eqs. (48), (49), (53), (54), (56) and (57) along with their boundary conditions (50), (55), (58) respectively are integrated using one correction

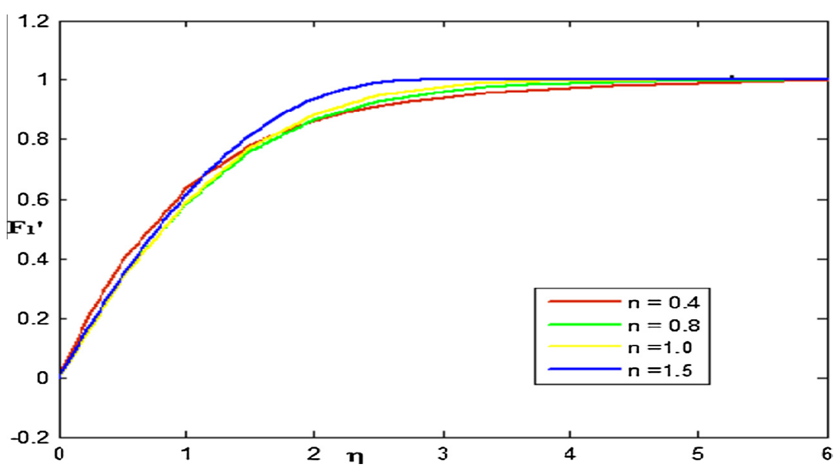

Figure 1 Graph of F_1' for shear thinning & shear thickening flows.

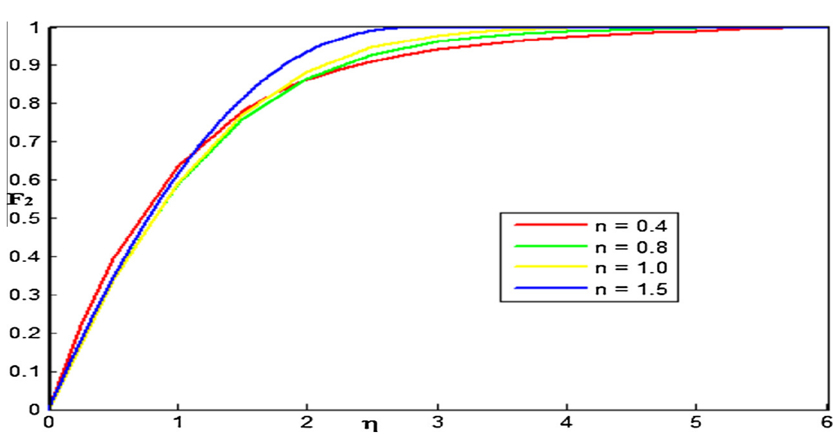

Figure 2 Graph of F_2 for shear thinning & shear thickening flows.

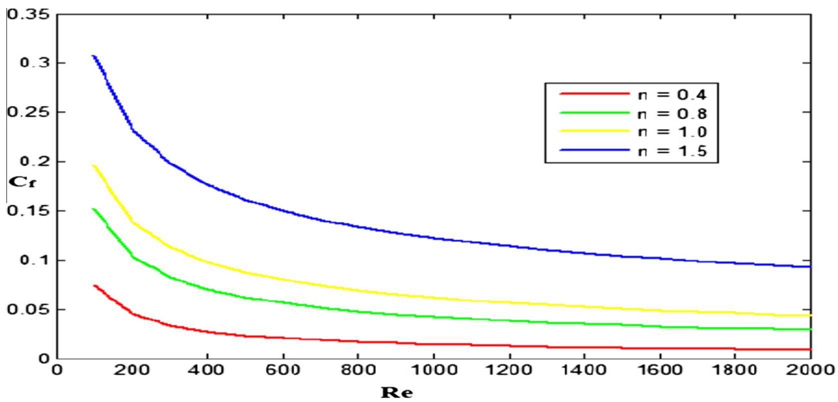

Figure 3 Variation in C_f for different Reynold number.

per step. The step size is taken as $2 * 10^{-4}$. Numerical integration is carried to a specific value of η. The correction Δx and Δz is determined from Eq. (51). The process is repeated until the relative change in correction is less than a small pre-assigned value. This value is taken as 10^{-8}. When this is achieved, the value of E is computed using Eq. (52). If E is greater than a small pre-assigned value, η is increased and the whole process is repeated. When E satisfies the test value, the calculations are stopped. Value of η at this point is taken as η_{edge}. The whole process is carried out for different values of power law index n. The velocity profiles obtained in each of these cases are presented graphically. The physical quantity, the coefficient of local skin friction C_f is given as follows

$$C_f = 2\left[F_1''(0)\right]^n R_e^{-\frac{1}{n+1}}$$

6. Conclusion

One parameter deductive group transformation has been applied to the considered governing equations and their boundary conditions, thereby successfully reducing two independent variables to one. All possible conditions under which the similarity solution for present flow situation exists, are automatically derived from the similarity requirement. Thus, the obtained similarity solution is in the most general form. Finally the coupled system of self-similar Eqs. (44) and (45) with boundary conditions (42) are solved numerically. The effects of velocity profiles in x and y directions are presented for the shear thinning (n = 0.4, 0.8), Newtonian (n = 1) and shear thickening (n = 1.5) of the quasi three dimensional fluid. It has been observed that the thickness of the viscous layer in the region close to the plate is higher for $n > 1$ compared to that when $n < 1$. Moreover, it is clear from Figs. 1 and 2 that the velocity components increase with the increase in the value of η (see Fig. 3).

It is interesting to note that Hansen and Na [5] have derived similarity solution of the considered equations using linear and spiral group of transformations whereas the similarity solution, here, has been derived using more general group theoretic method known as deductive group transformation [22] and has further been solved numerically based on MSABC. Moreover, Patel and Timol [23] have applied MSABC for two dimensional flows. The same technique has been extended for quasi three dimensional flows. From a practical point of view, similarity solutions might be applied to the study of boundary layer flow over aerodynamic configurations such as wings, missiles, fuselage forms or channel flows. Further work can be carried out on the different geometry of surfaces and co-ordinate systems associated with similarity analysis.

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